Some Inequalities for the Square Root of a Positive Definite Matrix*

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1. INTRODUCTION

Let A be a positive definite matrix, and let $A^{1/2}$ be the unique positive definite square root of A. Let $A \geqslant B$ denote that $A - B$ is nonnegative definite. Then we wish to demonstrate

THEOREM 1. *The following inequalities are valid:*

(a) *If* $A \geqslant B \geqslant 0$, *then* $A^{1/2} \geqslant B^{1/2}$.

(b) *If* $A, B > 0$, *then* $(\lambda A + (1 - \lambda B)^{1/2} \geqslant \lambda A^{1/2} + (1 - \lambda) B^{1/2}$, *for* $0 \leqslant \lambda \leqslant 1$.

(c) *If* $A, B \geqslant 0$, *then*

$$\left(\frac{A + B}{2}\right) \leqslant \left(\frac{A^2 + B^2}{2}\right)^{1/2} \leqslant \cdots \leqslant \left(\frac{A^{2^N} + B^{2^N}}{2}\right)^{1/2^N} \leqslant \cdots$$

for $N = 1, 2, \ldots$.

The results are certainly in the literature in various places. What is of interest is the method we employ based upon a direct representation of $A^{1/2}$. Establishing inequalities of the foregoing type is complicated by the fact that $A \leqslant B$ does not necessarily imply that $A^2 \leqslant B^2$; see [1].

* Dedicated to Professor A. M. Ostrowski on his 75th birthday.

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2. REPRESENTATION FOR $A^{1/2}$

Let (y, z) denote the inner product of two N-dimensional vectors y and z. Consider the problem of determining the minimum of the quadratic functional

$$J(x) = \int_0^T [(x', x') + (x, Ax)]\, dt, \tag{1}$$

where $x' = dx/dt$ and $A > 0$, over all vectors $x(t)$ such that $x' \in L^2(0, T)$ and $x(0) = c$. Then it is easy to show that the minimum exists and that

$$\min_x J(x) = (c, R(T)c), \tag{2}$$

where $R(T)$ is a positive definite matrix [2]. The theory of dynamic programming yields the Riccati differential equation

$$R' = A - R^2, \qquad R(0) = 0. \tag{3}$$

It is easy to see that $R(T)$ is monotone increasing in T and that

$$\lim_{T\to\infty} R(T) = A^{1/2}. \tag{4}$$

Hence, we have the representation

$$(c, A^{1/2}c) = \lim_{T\to\infty} \left[\min_x \int_0^T [(x', x') + (x, Ax)]\, dt \right]. \tag{5}$$

3. PROOF OF THEOREM 1(a, b)

From (5), the proof of the inequalities (a) and (b) follows easily. That of (a) is immediate; that of (b) follows from

$$\int_0^T [(x', x') + (x, (\lambda A + (1 - \lambda)B)x)]\, dt$$
$$= \lambda \int_0^T [(x', x') + (x, Ax)]\, dt + (1 - \lambda) \int_0^T [(x', x') + (x, Bx)]\, dt, \tag{6}$$

and the obvious result

$$\min_x \int_0^T [\cdots]\, dt \geqslant \lambda \min_x \int_0^T [\cdots]\, dt + (1-\lambda) \min_x \int_0^T [\cdots]\, dt. \tag{7}$$

4. PROOF OF THEOREM 1(c)

To establish (c), we begin with

$$\begin{aligned} (A-B)^2 &\geqslant 0, \\ A^2 + B^2 &\geqslant AB + BA. \end{aligned} \tag{8}$$

This is equivalent to the relation

$$\left(\frac{A+B}{2}\right)^2 \leqslant \left(\frac{A^2+B^2}{2}\right). \tag{9}$$

Applying Theorem 1(a) this yields

$$\left(\frac{A+B}{2}\right) \leqslant \left(\frac{A^2+B^2}{2}\right)^{1/2}. \tag{10}$$

Replacing A by A^2 and B by B^2, this yields

$$\left(\frac{A^2+B^2}{2}\right) \leqslant \left(\frac{A^4+B^4}{2}\right)^{1/2}. \tag{11}$$

Applying Theorem 1(a) again, we have

$$\left(\frac{A^2+B^2}{2}\right)^{1/2} \leqslant \left(\frac{A^4+B^4}{2}\right)^{1/4}, \tag{12}$$

and so on.

It is easy to see that

$$M(A, B) = \lim_{N\to\infty} \left(\frac{A^{2^N}+B^{2^N}}{2}\right)^{N/2} \tag{13}$$

exists and determines a matrix which belongs to the convex set of matrices which majorize both A and B.

5. INEQUALITIES FOR A^{-1}

Let us note in passing that the same method can be used to obtain inequalities for A^{-1}, starting with the representation

$$(c, A^{-1}c) = \max_x [2(x, c) - (x, Ax)], \tag{14}$$

for $A > 0$. From this follows the well-known result

$$A > B > 0 \text{ implies that } A^{-1} < B^{-1}, \tag{15}$$

and

$$(\lambda A + (1 - \lambda)B)^{-1} \leqslant \lambda A^{-1} + (1 - \lambda)B^{-1} \tag{16}$$

for $A, B > 0$, $0 \leqslant \lambda \leqslant 1$.

REFERENCES

1 R. Bellman, *Introduction to Matrix Analysis*, McGraw-Hill, New York, 1960.
2 R. Bellman, *Introduction to Modern Control Theory*, Academic Press, New York, 1968.

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